End of 19^{th}, beginning of 20^{th} century:

E&M was well established

Biggest problems in EM theory:

1. light is a wave of oscillating $\vec{E}$ & $\vec{B}$ fields
   ⇒ all known oscillations involve propagation of a disturbance

   e.g. water waves: waves travel along surface of water - water level is "disturbed" → no water, no waves

   e.g. sound: pressure disturbance in air → no air, no sound

   so what is being disturbed in the case of light?

   ⇒ guess: "ether" - invisible "fabric"

note: the "stiffer" the medium, the faster the propagation

ex: speed of sound in air: ~340 m/s
lead: 1210 m/s
aluminum: 6320 m/s
speed of light in "ether": 300,000,000 m/s

since $c = 3\times10^8$ m/s, ether should be extremely stiff! but it's invisible?

2. wave equation for EM waves in a vacuum w/no sources (starting w/ Maxwell's equations)

$$\frac{\partial^2 E}{\partial x^2} - \varepsilon_0\mu_0 \frac{\partial^2 E}{\partial t^2} = 0$$

this equation describes a propagating wave along x-direction:

$$E = E_0 \cos(kx - \omega t) \quad \text{where } \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = v$$

- This says c = constant which implies that there is some absolute reference frame where $v = 3\times10^8$ m/s of EM radiation
- Then you can measure your velocity in the ether by measuring speed of light in different directions

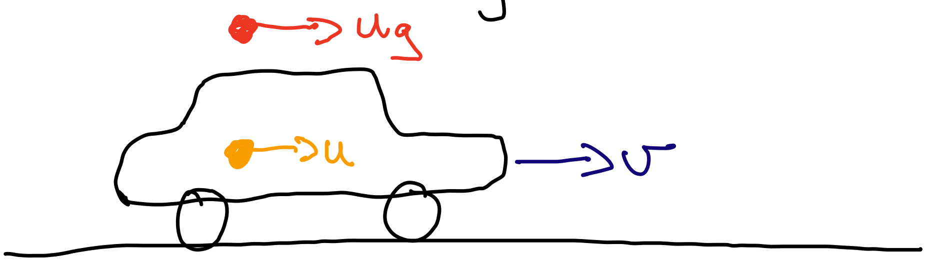
⇒ if ether exists then it would be the reference frame of absolute rest

⇒ ...or... if there was an absolute ref frame then that means there's an ether

Reference frame: the frame where you are <u>not</u> moving

ex: you drive in a car. The car is your reference frame. You pass me, I'm in the reference frame of the ground

let car have velocity v <u>relative to ground</u>



in the car someone throws a ball from back to front seat w/ velocity u

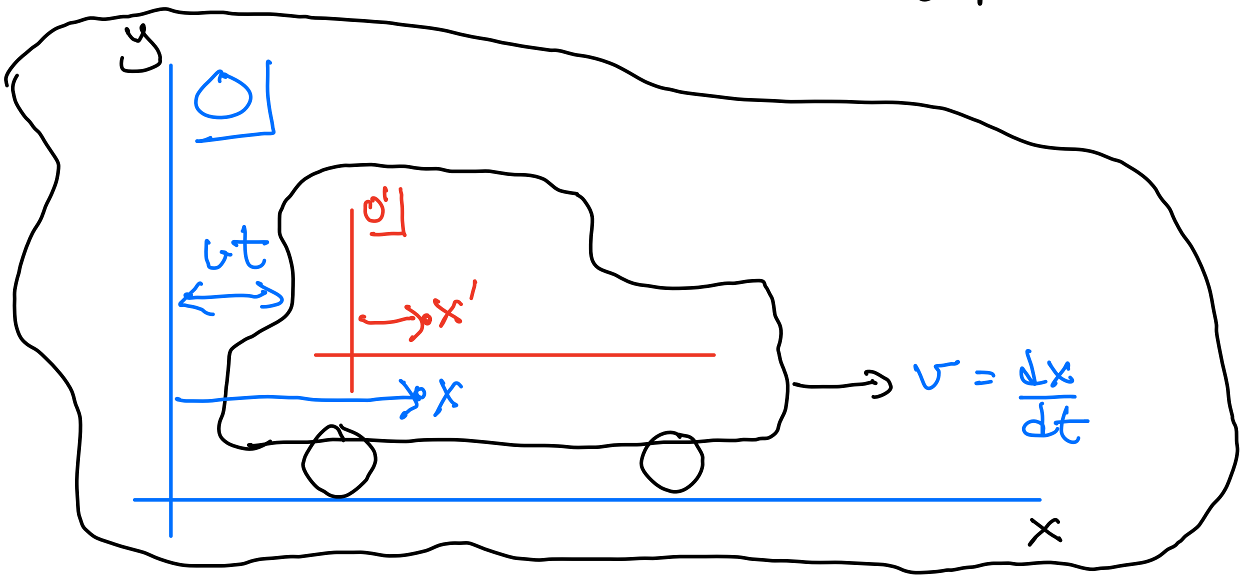
$\Rightarrow$ what is ball's velocity (u_g) relative to ground?

The answer is easy: $u_g = v + u$

More rigorous derivation: coordinate systems

Let O be reference frame of ground

and O' " " " in the car



$v = \frac{dx}{dt}$ where "x" is any coordinate position of the car in frame O

x' = coordinate position inside the car (relative to some agreed upon point of the car)

then x on ground is related to x' in the car by how far the car has moved in time t

$$x = x' + vt$$ "galilean" transformation

now calculate $u_g = \frac{dx}{dt}$ velocity of ball relative to ground

$$u_g = \frac{dx}{dt} = \frac{d}{dt}(x' + vt) = \underbrace{\frac{dx'}{dt}}_{\substack{\text{how position of ball} \\ \text{in car frame } O' \text{ is changing}}} + v = u + v$$

Problem w/ EM waves: if you shine light in the car forward and I measure its velocity on the ground, Galileo says I should get

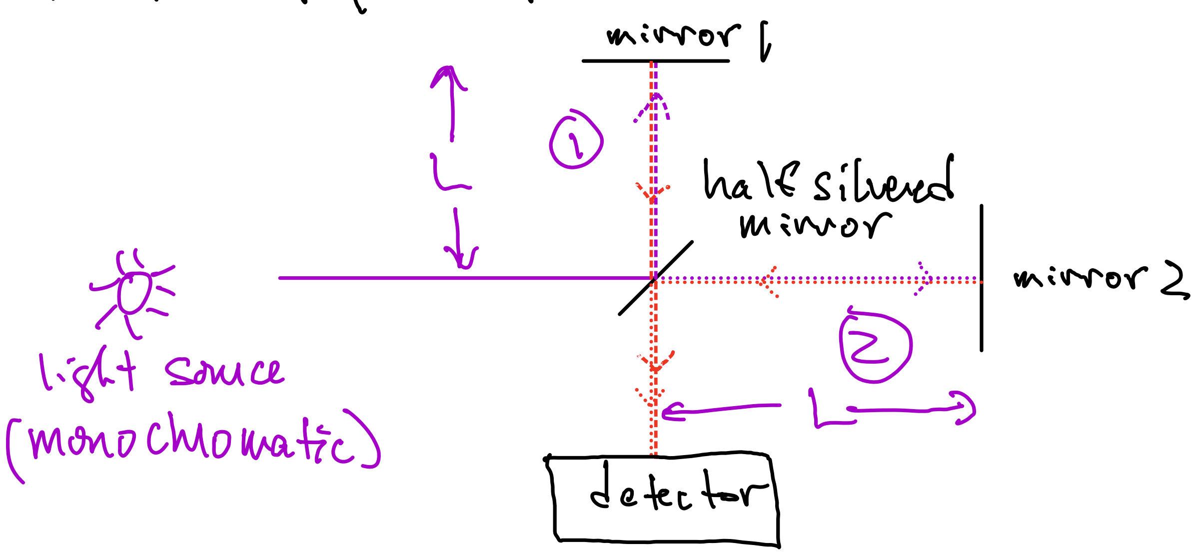
$$\underset{\substack{\text{speed of} \\ \text{light in} \\ \text{ground's frame}}}{u_{light}} = \underset{\substack{\text{speed of} \\ \text{light in car} \\ \text{frame}}}{c} + \underset{\substack{\text{speed of car} \\ \text{in ground's frame}}}{v}$$

so $u_{light} > c$ violates Maxwell's eqns

So to measure your velocity relative to the ether, you are measuring the absolute velocity inside the universe where $v=0$ ~~absolute~~

Michelson-Morley experiment
1887, Case Western university, Cleveland

Built interferometer:



- light from source hits a "half mirror" and splits beam into 2 parts

part 1. half light reflected up, then reflects back down by mirror 1 (dashes).

It then hits half mirror again and half goes thru to detector (ignore other half)

part 2. other half goes thru half mirror and is reflected back by mirror 2 (dotted).
It then hits half mirror again & half gets reflected down to detector

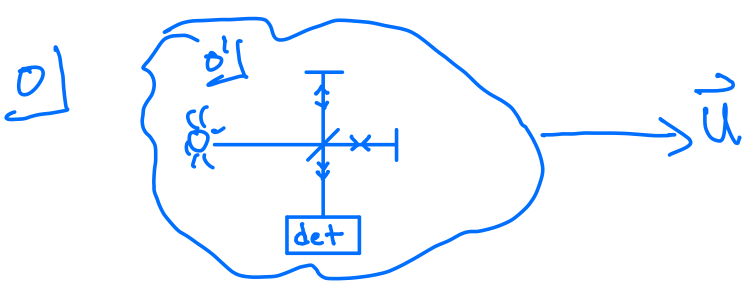
Make dist from half mirror to the 2 mirrors the same: L

Both beams interfere w/each other at detector.

If there were no ether then you could adjust the distances so that the 2 waves have a path difference $\Delta r = n\lambda$ where n is some number.

The waves would interfere constructively & you would see a bright spot.

Next say you were moving w/some velocity relative to the other along horizontal.



$x = x' + ut$

or $\Delta x = \Delta x' + u\Delta t$

t_1 = time for horizontal beam to go from half mirror to full mirror

t_2 = time from full mirror back to half

$t_1 > t_2$ because mirror is moving in the same direction as light

remember light moves at $v = c$ in ether only!!

Horizontal beam

- beam travels with vel = u in dir parallel to earth moving thru ether

- horizontal beam goes dist L in earth frame in time t_1

dist in ether is $c\Delta t_1 = L + u\Delta t_1$

on return from mirror to half splitter:

$$c\Delta t_2 = L - u\Delta t_2$$

light is anti parallel to $\vec{u}$

$\Delta t_1 > \Delta t_2$ because on 1st leg, destination (mirror) is moving away from starting pt (half mirror)

on return trip destination is moving towards starting point calculate total time in ether frame:

$$c\Delta t_1 = L + u\Delta t_1$$
$$(c-u)\Delta t_1 = L$$
$$\Delta t_1 = \frac{L}{c-u} = \frac{L}{c}\frac{1}{1-\beta} \qquad \beta \equiv u/c$$

$$\Delta t_2 = \frac{L}{c+u} = \frac{L}{c}\frac{1}{1+\beta}$$

here you see $\Delta t_1 > \Delta t_2$

total time is $\Delta t_1 + \Delta t_2 = \frac{L}{c}\left(\frac{1}{1-\beta} + \frac{1}{1+\beta}\right)$

$$= \frac{L}{c}\left(\frac{1+\beta+1-\beta}{1-\beta^2}\right) = \frac{2L}{c}\frac{1}{1-\beta^2}$$ let $\gamma^2 = \frac{1}{1-\beta^2}$

$$\Delta t_h = \frac{2L\gamma^2}{c}$$ "horizontal" time Δt_h

Vertical beam

This is the beam that is $\perp$ earth's motion thru ether

total distance is $2\sqrt{L^2+(u\Delta t)^2}$

this has to be $2c\Delta t$ in ether frame

so $2c\Delta t = 2\sqrt{L^2+(u\Delta t)^2}$

$$c^2\Delta t^2 = L^2 + u^2\Delta t^2$$
$$L^2 = (c^2-u^2)\Delta t^2$$
$$\Delta t = \frac{L}{\sqrt{c^2-u^2}} = \frac{L}{c}\frac{1}{\sqrt{1-\beta^2}} = \frac{L\gamma}{c}$$

total time $\Delta t_v = 2\Delta t = \frac{2L\gamma}{c}$

and $\Delta t_h = \frac{2L\gamma^2}{c}$

so $\Delta t_h > \Delta t_v$

let's say $\beta \ll 1$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = (1-\beta^2)^{-1/2} \sim (1+\tfrac{1}{2}\beta^2)$$ expand and keep smallest terms

$$\gamma^2 = \frac{1}{1-\beta^2} = (1-\beta^2)^{-1} \sim 1+\beta^2$$

$$\Delta t \equiv \Delta t_h - \Delta t_v = \frac{2L\gamma^2}{c} - \frac{2L\gamma}{c}$$
$$\sim \frac{2L}{c}\left(1+\beta^2-(1+\tfrac{1}{2}\beta^2)\right) \sim \frac{2L}{c}\frac{\beta^2}{2}$$
$$= \beta^2\frac{L}{c}$$

in that time, horizontal light goes dist

$$d = c\Delta t = \beta^2 L$$

light has wavelength ~ 500nm

if $d = \beta^2 L \cong 50\text{nm}$, you would see it in the interference pattern changing

for $L = 10\text{m}$ interferometer arm:

$d = 50 \times 10^{-9}\text{m} = \beta^2 \cdot 10\text{m}$

$$\beta^2 = 50 \times 10^{-10}$$

$$u = c * \sqrt{50 \times 10^{-10}} \cong 3 \times 10^8 \frac{m}{s} \cdot 7 \times 10^{-5}$$

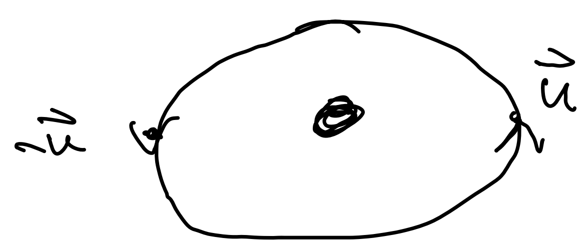
$$= 21 \times 10^3 \text{m/s}$$

$$\sim 2 \times 10^4 \text{m/s}$$

1. take interferometer & measure interference pattern, then adjust one full mirror distance so that you see a max

2. turn it at right angle - should see a big change

3. do the same 6 months later when on opposite side of sun

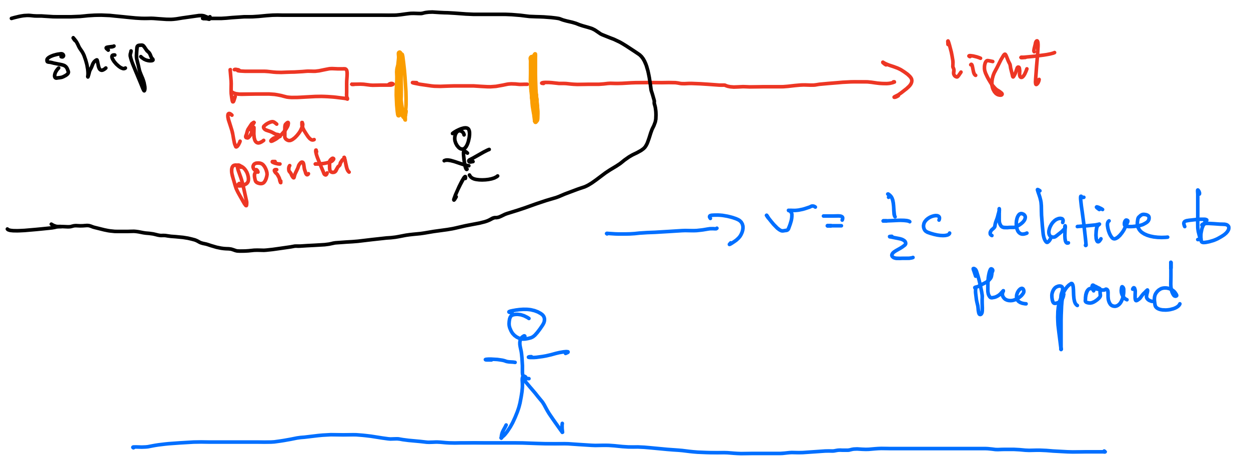


No effect seen!!!

No experiment ever tried has ever seen any ether.

So if the ether doesn't exist, then

1. There is no absolute ref frame. So no absolute velocities
   Conclusion: the laws of nature only depends on <u>relative</u> velocities
   more accurate: laws of nature are the same in all ref frames that have constant v ("inertial frames")
2. Maxwell's equations say $v_{light} = 3 \times 10^8 m/s$
   ⇒ but in which ref frame? all frames?
   this violates common sense!



- ship has a laser shining light along direction of motion.

  person on ship can measure light velocity by timing beam as it passes thru the two detectors.

  ⇒ will measure $v_{light} = c$ on ship

- person on ground can also measure light velocity w/ duplicate detectors on the ground

  will measure $v_{light} = c$ on ground!

  and <u>not</u> $v_{light} = c + v_{ship}$

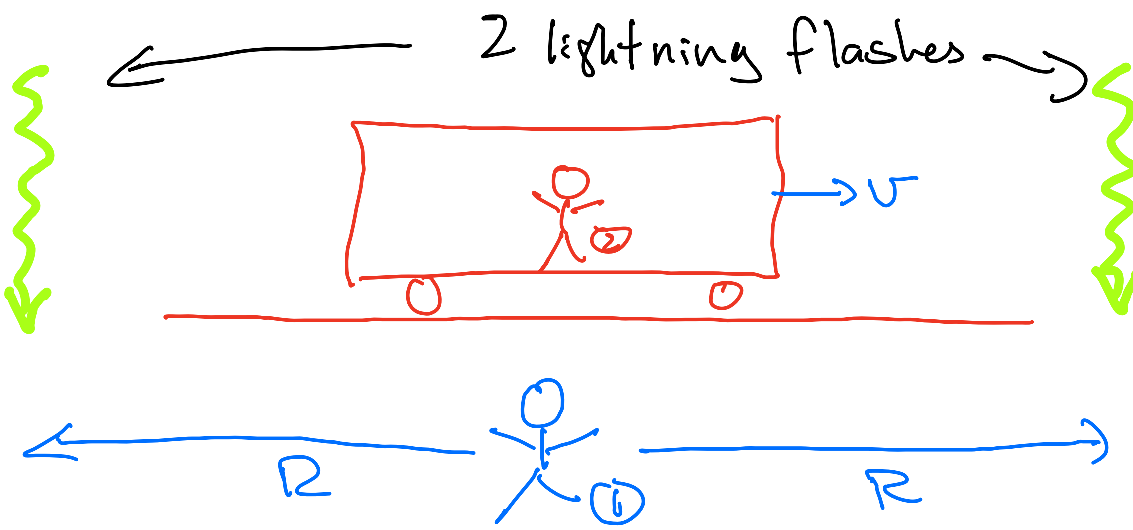
These two postulates are Einstein's Postulates

(#2 says that the speed of light is the largest possible <u>relative</u> velocity

How can both postulates be true <u>and</u> the Galilean transformation hold?

⇒ figuring this out is one of Einstein's many contributions

Simutaneity & time



person 1 on ground, person 2 on car w/vel. = v moving to right.

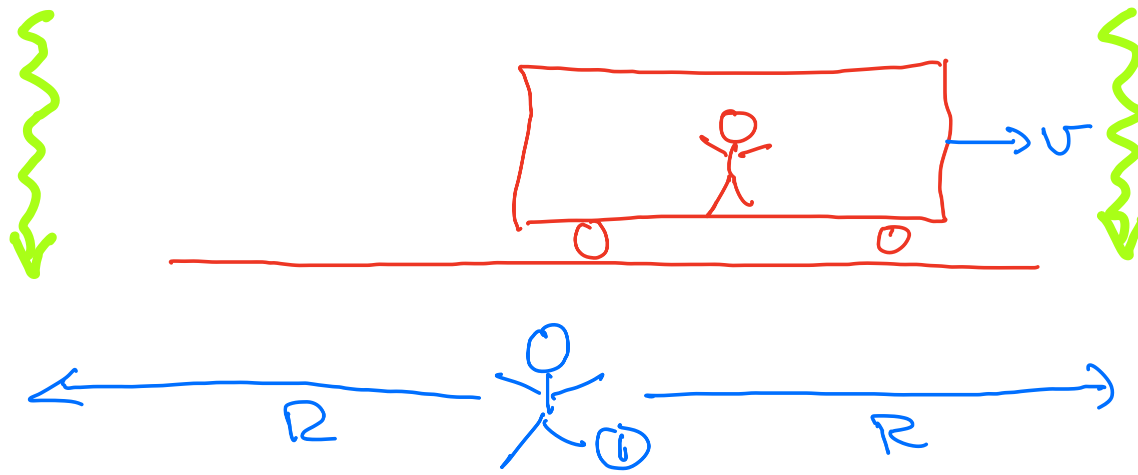
lightening strikes the ground the same distance R to left and right of person 1

1 says lightning bolts were simultaneous

because the light arrived at 1 at the same time

2 lightning flash took t_1 seconds to get to person 1

3 2 goes a distance $d_2 = v t_1$ in time t_1 and then sees 1st flash



2 sees the flash from the bolt on right 1^{st}, then the flash from the left next

so 2 disagrees that the 2 events (each flash is an event) were simultaneous (at the same time)

But – postulate 1 says all inertial reference frames are equivalent

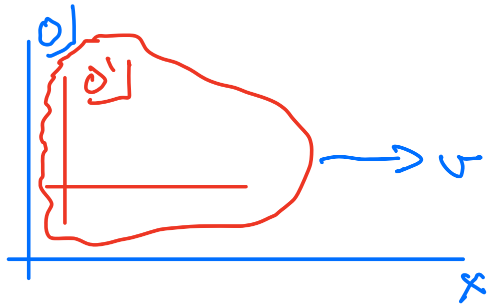
Therefore: simultaneity must not be anything fundamental

Therefore: time is <u>not</u> absolute!

$\Rightarrow$ implies that we have to modify Galilean transformation

O] is our ref frame

O'] moves w/ rel velocity v in O]



Galilean: $x = x' + vt$ needs adjustments

1. try $x = f(v)(x' + vt)$

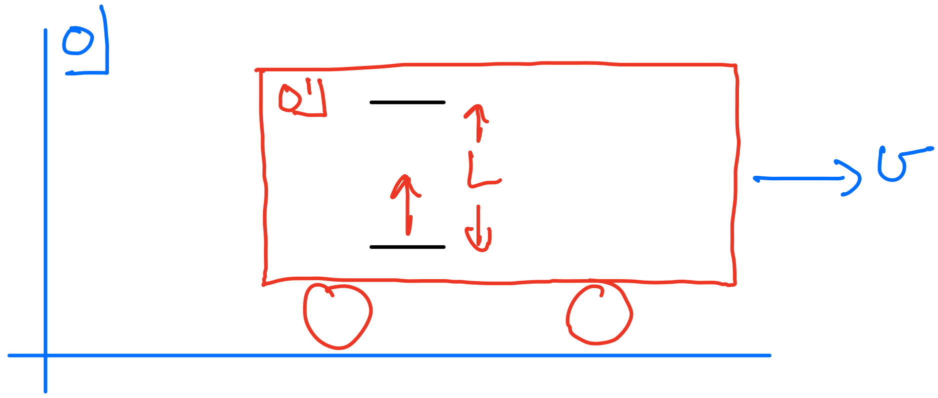
   where $f(v) \longrightarrow 1$ as $v \to 0$
   since Galilean does seem to work
   when $v \ll c$ (the "real" world)

2. since time is relative and <u>not</u> absolute, then time could be different in the 2 frames O] & O']

   so $x = f(v)(x' + vt')$ $\quad t \neq t'$

new questions:

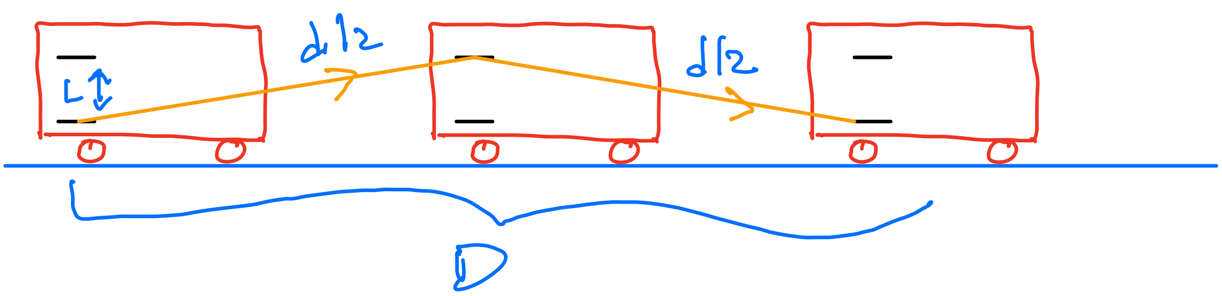
- how to calculate $f(v)$?
- how does time transform?



- 2 mirrors on train moving w/ velocity v in O]
- train is O'] frame
- bounce a beam of light between the mirrors

  $\Rightarrow$ takes $\Delta t'$ time to go distance $2L$ in train, velocity of light is c

  $\therefore \; 2L = c\Delta t'$
- in frame O], we see this



Distance train travels in time Δt: $D = v\Delta t$

Distance light travels along diagonal : $d = c\Delta t$

Distance light travels: $\left(\frac{d}{2}\right)^2 = \left(\frac{D}{2}\right)^2 + L^2$

and $d = c\Delta t$ since light travels at vel c in all frames
in ref frame O'

so $\left(\frac{d}{2}\right)^2 = \left(\frac{D}{2}\right)^2 + L^2$

$$\left(\frac{c\Delta t}{2}\right)^2 = \left(\frac{v\Delta t}{2}\right)^2 + \left(\frac{c\Delta t'}{2}\right)^2$$

rearrange, get rid of "2":

$c^2\Delta t^2 - v^2\Delta t^2 = c^2\Delta t'^2 \Rightarrow \Delta t' = \Delta t\sqrt{1 - \frac{v^2}{c^2}}$

define $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}} \Rightarrow \Delta t = \Delta t'\gamma$

note: O' is frame where events are happening at locations (x') that are not changing.

O' is the "proper frame" and

$\Delta t'$ is the "proper time" $\equiv \Delta\tau$

Proper time is the time in the frame where position isn't changing

ex: you are on an airplane and hold your breath for 60 sec.

in your frame, position isn't changing

so proper time $\Delta\tau = 60$ sec

then a time interval in some frame moving w/ velocity v relative to proper frame is Δt and

$$\boxed{\Delta t = \gamma \Delta\tau} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \ \beta \equiv v/c$$

$\Delta\tau$ is always the shortest time interval than the interval in any other frame

This is called $\boxed{\text{"time dilation"}}$

ex: you are moving w/ velocity v relative to me. a year goes by in your frame

$$\Delta\tau = 1 \text{ year}$$

what do I measure?

$$\Delta t = \gamma \Delta\tau$$

if $v = 10{,}000$ mph:

$$v = 10^4 \frac{mi}{hr} * \frac{5280ft}{mi} * \frac{1\,m}{3.28ft} * \frac{1\,hr}{3600s}$$

$$= 10^4\, mph * 0.45 \frac{m/s}{mph} = 4500\, m/s$$

$$\beta = \frac{v}{c} = \frac{4500}{3\times10^8} = 1.5\times10^{-9}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = (1-\beta^2)^{-1/2} \approx 1 + \frac{1}{2}\beta^2 \sim 1$$

what if $\beta = 0.1$? ($v = 3\times10^7\, m/s \approx 18{,}600\, mi/s$!)

$$\gamma = \frac{1}{\sqrt{1-0.1^2}} = 1.005$$

$$\Delta t = 1.005\, \Delta\tau = 1.005\, yr$$

$$0.005\, yr * \frac{365\, d}{yr} = 1.825 \text{ days!}$$

ex: a muon is a particle that decays on average after 2.2μs

muons are made constantly in the upper atmosphere when cosmic rays hit the atmosphere. They are created with velocities:

$\beta = 0.99$ wrt earth frame

How long do the muons live in earth's

frame?

$\Delta\tau = 2.2\mu s$ in muon's rest frame (where it is standing still)

$$\Delta t_{earth} = \gamma \Delta\tau = \frac{2.2\mu s}{\sqrt{1-0.99^2}} = 7.09 \times 2.2\mu s$$
$$= 15.6\mu s$$

in $15.6\mu s$ muons travel distance (in earth's frame)

$$D = vt = .99c \times 15.6\mu s$$
$$= 0.99 \times 3 \times 10^8 \frac{m}{s} \times 15.6 \times 10^{-6} s$$
$$= 4631 m = 4.6 km \sim 3 \text{ miles}$$

"Twin paradox"

Twins Alpha & Beta, same age
Alpha stays on earth.
Beta flies away on spaceship, $\beta = 0.6$ towards nearest star α-Centauri, 5 lt-yrs away

Alpha : 

Beta : 

$$\beta = 0.6 \text{ so } \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.6^2}} = 1.25$$

Alpha frame (earth): Beta goes 5 lt-yrs at $\beta = 0.6$

$$d = v \cdot \Delta t_\alpha = \frac{v}{c} * (\Delta t_\alpha \cdot c)$$

note: $c \cdot \Delta t$ is distance light travels in time Δt

$\frac{v}{c} = \beta$ dimensionless number

so if we measure distances in light-time (e.g. light-years) and time in years then $v \equiv \beta$

so $v = 0.6$ and d = light-years so Δt_α = years

$$\Delta t_\alpha = \frac{d}{v} = \frac{5 \text{ lt-years}}{0.6} = 8.3 \text{ years}$$ time interval in Alpha frame

Alpha measure's Beta clock time:

proper time in Beta frame

$$\Delta t_\alpha = \gamma \Delta t_\beta$$

$$\therefore \Delta t_\beta = \frac{\Delta t_\alpha}{\gamma} = \frac{8.3 \text{ yr}}{1.25} = 6.7 \text{ yrs}$$

Beta turns around and heads home at same velocity

total time in Alpha (earth) frame:

$$\Delta t_\alpha = 2 * 8.3 = 16.6 \text{ yrs}$$

total time in Beta (ship) frame:

$$\Delta t_\beta = 2 * 6.7 = 13.4 \text{ yrs}$$

Alpha is now older than Beta!

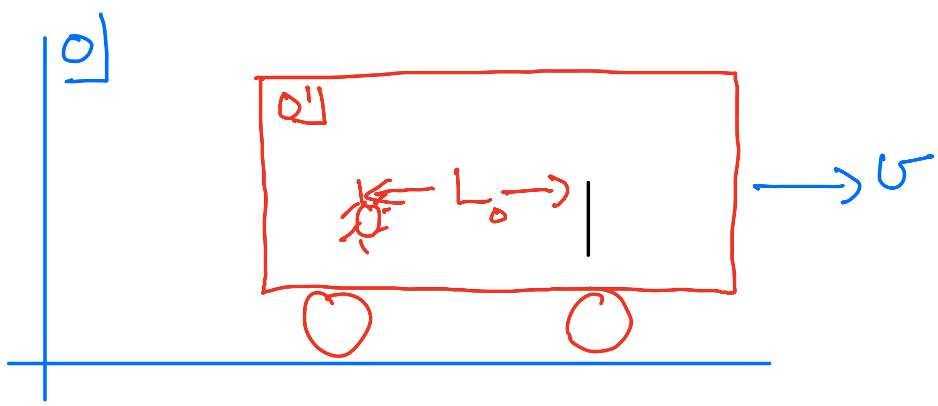
Relativity lets you go into the future!

Why does this work? Since velocity is relative and the journey is symmetric why a different age?

Because Beta had to accelerate at some point. That makes the 2 frames unequal, not symmetric!

Time dilation covers time intervals

$\Rightarrow$ what about length intervals



shine a light in proper frame, it hits a mirror a distance L away and bounces back so $c\Delta t' = 2L_0$ $\quad \Delta t'$ is transit time in O'

in frame O the time to go from source to ruler is measured to be Δt_1 and the distance is L which maybe is <u>not</u> the same as L_0!

total distance light traveled in O to mirror

$$d_1 = L + v\Delta t_1 = c\Delta t_1$$

so $L = (c-v)\Delta t_1$

or $\Delta t_1 = \dfrac{L}{c-v}$

on return trip, mirror to source, measured Δt_2

and $d_2 = L - v\Delta t_2 = c\Delta t_2$

so $\quad \Delta t_2 = \dfrac{L}{c+v}$

then $\Delta t = \Delta t_1 + \Delta t_2 = \dfrac{L}{c-v} + \dfrac{L}{c+v}$

$$= \frac{L(c+v+c-v)}{(c-v)(c+v)} = \frac{2Lc}{c^2-v^2}$$

$$\Delta t = \frac{2L}{c}\frac{1}{1-v^2/c^2} = \frac{2L}{c}\gamma^2$$

$$\Delta t' = \frac{2L_0}{c}$$

and we know that $\Delta t = \Delta t' \gamma$ ($\Delta t'$ = proper time)

so $\frac{2L\gamma^2}{c} = \left(\frac{2L_0}{c}\right)\gamma$

$$\boxed{L = \frac{L_0}{\gamma}}$$ $L < L_0$!

This is called Lorentz contraction or
length "

ex: a spaceship goes at $\beta = 0.9$ and is 100m long. This means in proper frame of the spaceship, it is measured to be 100m ($L_0 = 100m$)

what is the length on earth's frame where it goes at $\beta = 0.9$?

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.9^2}} = 2.3$$

on earth $L = \frac{L_0}{\gamma} = \frac{100}{2.3} = 43.6m$!

how can this be?

It's because of the relativity of simultaneity. On earth as the ship passes your 100m ruler, you record the position of the front and back of the ship AT THE SAME TIME !!

But on the ship, they will say you measured the front ~~1st~~, then the back.

But this means that you could have the 100m ship in a room with front & rear doors closed at same time (in your frame) and it will fit!

How to use this?

Galilean: $x = x' + vt'$
$t = t'$ } this is very accurate!

Relativistic transformation has to reduce to Galilean when $v \to 0$

try $x = f(v)(x' + vt')$
and $f(v) = f(-v)$ and
$f(0) = 1$

if you reverse frames, then O goes at vel $-v$ with respect to O'

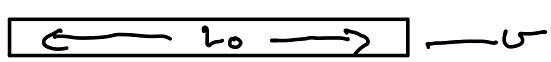
and $x' = x - vt$

so try $x' = f(v)(x - vt)$ relativistic

This also works for intervals:

$$\Delta x' = f(v)(\Delta x - v\Delta t)$$

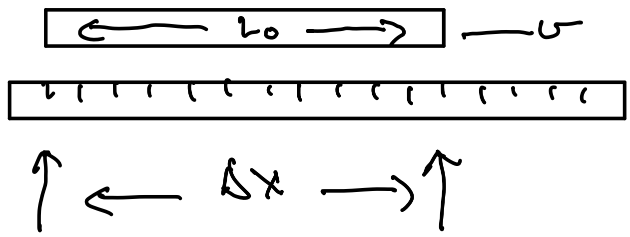
Now we measure the length of a moving object:



it has length L_0 in the proper frame

$$\Delta x' = L_0$$

in O, we measure the length by recording the beginning & ends of the moving object with a stationary ruler



we get a length $L = \Delta x$

when we record the ruler values, we do so simultaneously at both ends

so $\Delta t = 0$ between the events (recording)

transforming back:

$$\Delta x' = f(v)(\Delta x - v\Delta t)$$

but $\Delta t = 0$

so $\Delta x' = f(v)\Delta x$

$$L_0 = f(v)L = f(v)\frac{L_0}{\gamma}$$

so $f(v) = \gamma$

so the coordinate transformations are:

$$x = \gamma(x' + vt') = \gamma(x' + \beta ct')$$

$x' = \gamma[x - \beta c t]$
$y' = y$ (perpendicular to $\vec{v}$)

These are relativistic transformations of position

For previous ruler example:

$$\Delta x = \gamma(\Delta x' + \underbrace{\beta c}_{v} \Delta t')$$

$\Delta x = \frac{L_0}{\gamma}$, $\Delta x' = L_0$

so $\frac{L_0}{\gamma} = \gamma(L_0 + \beta c \Delta t')$

$$L_0\left(\frac{1}{\gamma} - \gamma\right) = \gamma \beta c \Delta t'$$

$\Delta t' = \frac{L_0\left(\frac{1}{\gamma} - \gamma\right)}{\gamma \beta c}$ that's what a person in ship will say is the time diff between the 2 measurements in O

$$\frac{\frac{1}{\gamma} - \gamma}{\gamma} = \frac{1}{\gamma^2} - 1 = (1-\beta^2) - 1 = -\beta^2$$

so $\Delta t' = -\frac{L_0 \beta}{c}$

or $c\Delta t' = -L_0 \beta$

note $\Delta t' = t_2 - t_1$ so $\Delta t' < 0$ means t_2 is before t_1

how does time transform?

back to $\Delta x' = \gamma(\Delta x - v\Delta t)$

then $\Delta x = \frac{\Delta x'}{\gamma} + v\Delta t$

to transform to other frame switch ' and make $v \to -v$ and swap "prime"

then $\Delta x' = \frac{\Delta x}{\gamma} - v\Delta t'$

$= \gamma(\Delta x - v\Delta t)$ from eqns

so $\frac{\Delta x}{\gamma} - v\Delta t' = \gamma(\Delta x - v\Delta t)$

$$\Delta x\left(\frac{1}{\gamma} - \gamma\right) + \gamma v \Delta t = v\Delta t'$$

$$\frac{1}{\gamma} - \gamma = \frac{1-\gamma^2}{\gamma} = \frac{1 - \frac{1}{1-\beta^2}}{\gamma} = \frac{1-\beta^2-1}{\gamma\underbrace{(1-\beta^2)}_{\frac{1}{\gamma^2}}} = -\gamma\beta^2$$

so $-\gamma\beta^2\Delta x + \gamma v\Delta t = v\Delta t'$

$v = \beta c$ so $\beta c \Delta t' = \gamma\beta c\Delta t - \gamma\beta^2\Delta x$

$$c\Delta t' = \gamma(c\Delta t - \beta\Delta x)$$

This is full Lorentz transformation:

$$\boxed{\begin{aligned} \Delta x' &= \gamma(\Delta x - \beta c \Delta t) \\ c\Delta t' &= \gamma(c\Delta t - \beta \Delta x) \end{aligned}}$$

and swapping frames:

$$\boxed{\begin{aligned} \Delta x &= \gamma(\Delta x' + \beta c \Delta t') \\ c\Delta t &= \gamma(c\Delta t' + \beta \Delta x') \end{aligned}}$$

note:

$$(c\Delta t)^2 - (\Delta x)^2 = \gamma^2 (c\Delta t' + \beta \Delta x')^2 - \gamma^2 (\Delta x' + \beta c \Delta t')^2$$

$$= \gamma^2 \left[(c\Delta t')^2 + 2\beta c \Delta t' \Delta x' + \beta^2 \Delta x'^2 - \Delta x'^2 - 2\beta c \Delta t' \Delta x' - \beta^2 c^2 \Delta t'^2 \right]$$

$$= \gamma^2 \left[(c\Delta t')^2 \underbrace{(1-\beta^2)}_{\frac{1}{\gamma^2}} + \Delta x'^2 \underbrace{(\beta^2 - 1)}_{-\frac{1}{\gamma^2}} \right]$$

$$= (c\Delta t')^2 - (\Delta x')^2$$

so $(c\Delta t)^2 - (\Delta x)^2$ is invariant same in all frames ex: take 2 "events" in frame O

event 1: at x_1, t_1

" 2: at x_2, t_2

$\Delta x = x_2 - x_1$

$\Delta t = t_2 - t_1$

then in any frame moving w/velocity v in frame O (the O' frame)

$$(c\Delta t')^2 - (\Delta x')^2 = (c\Delta t)^2 - (\Delta x)^2$$

this is called an "invariant"

$\Rightarrow$ same value in <u>all</u> reference frames

(as long as they are inertial, v = constant)

Special Relativity:

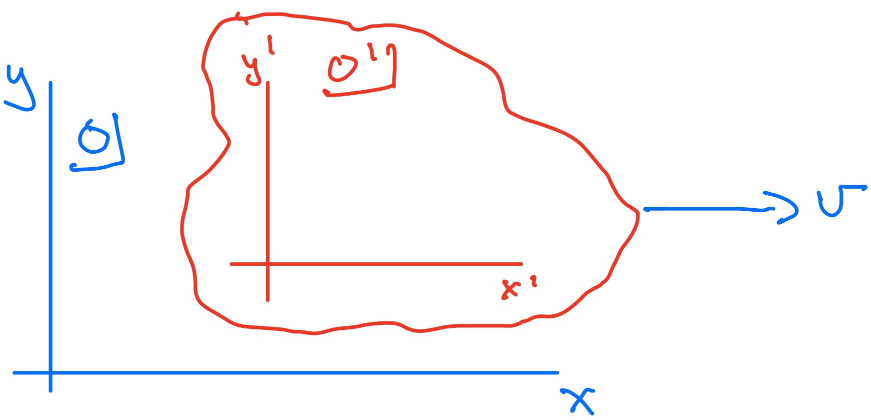
relativity $\Rightarrow$ only relative velocities matter and there is no absolute velocity

special $\Rightarrow$ velocity is constant

Lorentz coordinate transformations

relative velocity is along x direction

z is out of page



Galileo: transform from O' to O frame:

$$x = x' + vt$$

$$\left.\begin{aligned} y &= y' \\ z &= z' \end{aligned}\right\} \perp \text{ to direction of motion}$$

$$t = t'$$

to transform from O to O'

$$x' = x - vt$$

$$\left.\begin{aligned} y' &= y \\ z' &= z \end{aligned}\right\} \perp \text{ to direction of motion}$$

$$t' = t$$

Relativistic, O' to O

$$x = \gamma(x' + \beta ct')$$
$$y = y'$$
$$z = z'$$
$$ct = \gamma(ct' + \beta x')$$

O to O'

$$x' = \gamma(x - \beta ct)$$
$$y' = y$$
$$z' = z$$
$$ct' = \gamma(ct - \beta x)$$

note:
- write time as ct so it has same units as space $\Rightarrow$ 4-D space-time
- motion along x "mixes" space & time
- for $\beta \to 0$, $\gamma \to 1$, reduces to Galilean

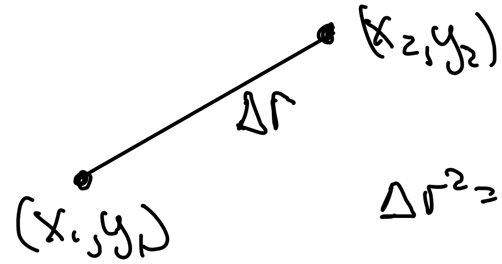
4-vector:

in space, 3-vector $\vec{r} = (x, y, z)$

now we are in 4-D so 4-vector will be

$$x = (ct, \vec{r}) \text{ shorthand for } (ct, x, y, z)$$

an "event" is a 4-D coordinate (space & time) in 3-D space, dist between coordinates is invariant w/respect to coordinate transformations (pick any coordinate system, same Δr)



$$\Delta r^2 = \Delta x^2 + \Delta y^2$$

in 4-D space-time the invariant is:

$$\Delta R^2 = c^2 \Delta t^2 - \Delta r^2$$
$$= c^2 (\Delta t')^2 - (\Delta r')^2$$

or any other coordinate

note: an "event" is something that happens at a position x & time t

we write $E_1 = (ct_1, x_1)$ coordinates

$\Rightarrow$ for 2^{nd} event we have $E_2 = (ct_2, x_2)$

then $\Delta R^2 = (c\Delta t)^2 - x^2$ is invariant

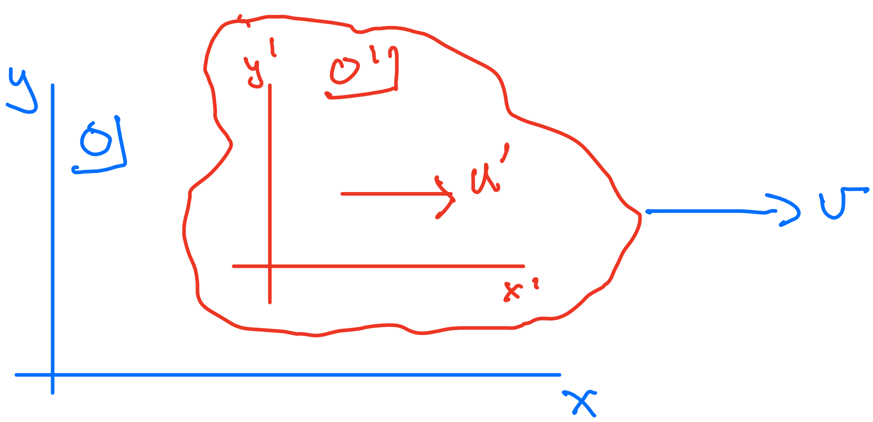
in the proper frame, a time interval $\Delta\tau$ happens at a fixed place

so $\Delta R^2 = (c\Delta\tau)^2 = ((c\Delta t')^2, (\Delta x')^2)$

so $\Delta R^2 =$ proper time interval!

Velocity transformation

- frame O'] moves w/ vel v in frame O]
- in O'] something moves along direction of motion w/ velocity u'



ex: you are in an airplane moving w/ vel v and you throw a ball down the aisle w/ velocity u' in the airplane's frame

⟹ what does someone on the ground measure? call that velocity u

x' marks the position of the ball

$$x = \gamma(x' + \beta c t')$$
$$ct = \gamma(ct' + \beta x')$$

and $u = \frac{dx}{dt}$

$$dx = \gamma(dx' + \beta c\, dt')$$
$$c\,dt = \gamma(c\,dt' + \beta dx')$$

then $$\frac{dx}{c\,dt} = \frac{dx' + \beta c\, dt'}{c\,dt' + \beta dx'} \cdot \frac{1/dt'}{1/dt'}$$
$$= \frac{\frac{dx'}{dt} + \beta c}{c + \beta \frac{dx'}{dt}}$$

$$u' = \frac{dx'}{dt}$$

so $$\frac{dx}{c\,dt} = \frac{u' + \beta c}{c + \beta u'}$$

or $$u = \frac{u'c + \beta c^2}{c + \beta u'} = \frac{u' + \beta c}{1 + \beta u'/c}$$

$\beta c = v$

so $$\boxed{u = \frac{u' + v}{1 + \frac{v u'}{c^2}}}$$

check: if instead of throwing a ball w/ vel u' we shine a light, then $u' = c$

$$u = \frac{c + v}{1 + v/c} = \frac{c(1 + v/c)}{1 + v/c} = c \quad \text{yes!}$$

velocity of light is the same in <u>both</u> frames

ex: spaceship goes at $v = 0.8c$ in earth frame
it throws a probe forward at $u' = 0.5c$

in earth's frame $u = \dfrac{u' + v}{1 + u'v/c^2}$

$$= \left(\frac{0.5 + 0.8}{1 + 0.5 \times 0.8}\right) c$$

$$= 0.93c$$

now take $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and unravel:

$$\gamma^2 = \frac{1}{1-\beta^2} \Rightarrow \gamma^2 - \gamma^2\beta^2 = 1$$

$\beta = v/c$ so this is $\gamma^2 c^2 - \gamma^2 v^2 = c^2$

this looks like an invariant!

(remember $(ct)^2 - x^2$ = same value in all frames

if we define 4-velocity like this

$$V = (\gamma c, \gamma \vec{v})$$

then the invariant is c^2

now multiply by m_0, the mass of an object

$m_0 \equiv$ rest mass $\rightarrow$ mass as measured in the proper frame

then $P = m_0 V = (m_0 \gamma c, m_0 \gamma \vec{v})$

this looks like the 4-D analog of momentum

invariant is $m_0^2 c^2$:

$$m_0^2 \gamma^2 c^2 - m_0^2 \gamma^2 v^2 = m_0^2 c^2$$

let $E = m_0 \gamma c^2$ relativistic energy

$p = m_0 \gamma v$ " momentum

then the invariant is

$$(m_0 \gamma c)^2 \cdot c^2 - (m_0 \gamma v)^2 c^2 = (m_0 c)^2 c^2$$

$$\underbrace{(m_0 \gamma c^2)^2}_{E^2} - \underbrace{(m_0 \gamma v)^2}_{p^2} c^2 = (m_0 c^2)^2$$

so $\boxed{E^2 - p^2 c^2 = (m_0 c^2)^2}$

notice $m_0 c^2$ is independent of velocity!

write $E = \sqrt{p^2 c^2 + (m_0 c^2)^2} = m_0 c^2 \sqrt{1 + \left(\frac{pc}{m_0 c^2}\right)^2}$

the term $\frac{pc}{m_0 c^2}$ is <u>always</u> small except when $\beta \to 1$ $(p = \gamma m_0 v)$

so expand: $(1 + x^2)^{1/2} \sim 1 + \frac{x^2}{2}$

so $E \Rightarrow m_0^2 c^2 \left(1 + \left(\frac{pc}{2 m_0 c^2}\right)^2\right)$

$$= m_0 c^2 + \frac{p^2 c^2}{2 m_0 c^2}$$

$$= m_0 c^2 + \frac{p^2}{2 m_0^2}$$

$$\frac{p^2}{2m_0} = \frac{m_0^2 v^2}{2m_0} = \frac{1}{2}mv^2 \Rightarrow \text{Kinetic energy}$$

so $\underbrace{E = m_0c^2}_{\text{rest mass energy}} + \underbrace{KE}_{\text{kinetic}}$

from above: $E = m_0\gamma c^2$ so we write $m = \gamma m_0$ to get famous formula

$$\boxed{E = mc^2}$$

What does this formula mean?

- when $\gamma = 1$ ($\beta = 0$), $E = m_0c^2$

  this is called rest-energy

  All particles that have mass have an "internal" rest energy even if they are not moving (in their proper frame)

Ex: 1 gm particle at rest

$$E = mc^2 = 10^{-3}\,\text{kg} \cdot (3 \times 10^8\,\text{m/s})^2$$

$$= 9 \times 10^{13}\,\text{J} \Leftarrow \text{enormous!}$$

note: 1 BTU = 1055 J so 1 gram contains:

$$E = 9\times10^{13}\,J * \frac{1\,BTU}{1055\,J} = 8.5\times10^{10}\,BTU$$

$= 0.085$ Trillion BTU

in 2018, state of MD used 1400 Trillion BTU

so $14\ T\ BTU * \frac{1\,g}{.085\,TBTU} = 165\,gm \sim \frac{1}{3}$ pound!

$\frac{1}{3}$ pound of mass contains enough mass-energy to power all of MD for a year

The rest mass energy is the energy of something in the proper frame

- if a mass m_0 moves w/ velocity v in $O|$

  $E = m_0c^2$ in proper frame $O'|$

  $\beta = v/c$ & $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ vel of $O'|$ in $O|$

  energy in $O|$ is given by

  $$E^2 = (m_0c^2) + (pc)^2$$

  where $E = \gamma m_0 c^2$ & $p = \gamma m v$

if $pc \ll m_0c^2$ then can write

$$E = m_0c^2 + KE \qquad KE = \frac{1}{2}mv^2$$

as usual

note: some physicists say $E = mc^2$

where $m = \gamma m_0$ is the mass

Then when you add energy, it speeds up and γ increases.

Does this mean mass increases?

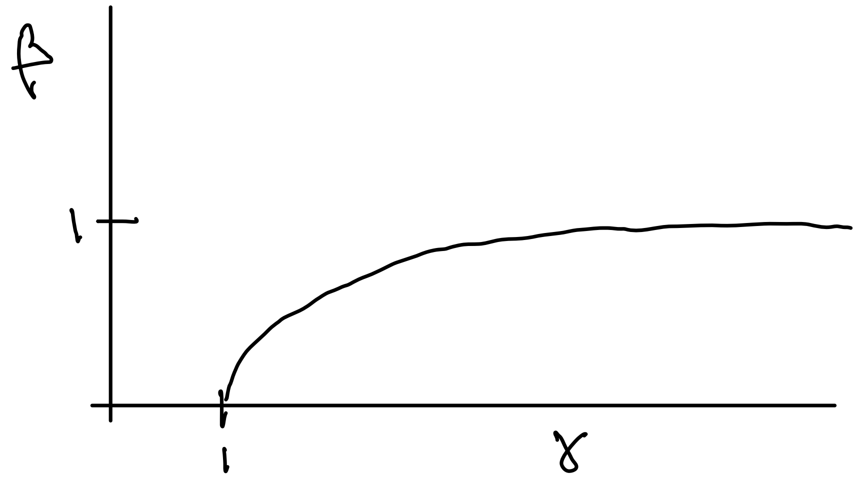
Well, it's true that $E = m_0c^2$ and if you were on O and measured mass of particle in O' you would measure

$$m = \gamma m_0$$

But mass isn't really increasing. γ is increasing!

so as you add energy, γ increases but velocity increases slowly and will <u>never</u> get exactly to $\beta = 1$

Here is a plot of β vs γ



$p = \gamma m_0 v = \gamma \beta m_0 c$ and $E = \gamma m_0 c^2$

$E^2 = (pc)^2 + (m_0 c^2)^2$

$(\gamma m_0 c^2)^2 = (\gamma \beta m_0 c^2)^2 + (m_0 c^2)^2$

cancel out m_0^2 everywhere:

$\gamma^2 c^4 = \gamma^2 \beta^2 c^4 + c^4$

divide by c^4:

$\gamma^2 = \gamma^2 \beta^2 + 1$

$\gamma^2 (1 - \beta^2) = 1$

$\gamma^2 = \dfrac{1}{1-\beta^2}$ ✓

ex: electron mass $= 9.109 \times 10^{-31}$ kg rest mass

$$m_e = 9.109 \times 10^{-31} \text{ kg}$$

$$m_e c^2 = 9.109 \times 10^{-31} \text{ kg} * \left(3 \times 10^8 \frac{m}{s}\right)^2$$

$$= 8.198 \times 10^{-14} \text{ J}$$

remember previous chapter, eV electron volt as a measure of energy:

a charged particle w/charge q going thru a potential change ΔV will gain ("downhill") or lose ("uphill") an amount of energy $E = q\Delta V$

if $q = +1.6 \times 10^{-19}$ C & $\Delta V = 1$ volt then

$$E = 1.6 \times 10^{-19} \text{ C} \cdot 1 \text{V} = 1.6 \times 10^{-19} \text{ J}$$

we can define $\underline{1 \text{eV} = 1.6 \times 10^{-19} \text{ J}}$

so rest energy of electron will be

$$E_0 = m_0 c^2 = 8.198 \times 10^{-14} \text{ J} * \frac{1 \text{eV}}{1.6 \times 10^{-19} \text{ J}}$$

$$= 511 \times 10^3 \text{ eV}$$

10^3 eV = 1 keV thousand

10^6 eV = 1 MeV million

$10^9 eV = 1 GeV$ billion

can write $E_0 = 0.511 MeV$ for electron rest energy

in any relativity problem we <u>always</u> see m_0 with c or c^2

$$E_0 = m_0 c^2$$
$$E = mc^2 \quad (m = \gamma m_0)$$
so can write $E = \gamma E_0$

$$p = m_0 \gamma v = m_0 c \gamma \beta$$
so $pc = m_0 c^2 \gamma \beta = \gamma \beta E_0$

trick: use eV for masses & energy & momentum!

ex: electron has $v = 2.5 \times 10^8$ m/s

$$\beta = \frac{2.5 \times 10^8}{3 \times 10^8} = 0.833$$

$$\gamma = \sqrt{\frac{1}{1-\beta^2}} = 1.81$$

$$p = m_0 v \gamma = 9.109 \times 10^{-31} kg * 2.5 \times 10^8 \frac{m}{s} * 1.81$$
$$= 4.12 \times 10^{-22} \ kg\,m/s$$

$$E = m_0 c^2 \gamma = 8.198 \times 10^{-14} J \cdot 1.81 = 1.48 \times 10^{-13} J$$

now convert to eV

$$pc = 4.12\times10^{-22} * 3\times10^{8} = 1.24\times10^{-13}\,J$$

$$= 1.24\times10^{-13}\,J * \frac{1\,eV}{1.6\times10^{-19}} = 7.72\times10^{5}\,eV$$

$$= 0.77\,MeV \quad \text{so} \quad p = 0.77\,MeV/c$$

$$E = 1.48\times10^{-13}\,J * \frac{1\,eV}{1.6\times10^{-19}\,J} = 9.27\times10^{5}\,eV$$

$$= 0.93\,MeV$$

no do the proplem in eV from start:

$$p = \gamma m_0 v = \gamma m_0 c^2 \cdot \frac{v}{c^2} = \gamma \underbrace{(m_0c^2)}_{0.511\,MeV} \beta / c$$

$$= 1.81 * 0.511\,MeV * 0.83/c$$

$$= 0.77\,MeV/c$$

$$E = \gamma m_0 c^2 = 1.81 \times 0.511\,MeV = 0.93\,MeV$$

voila!

ex: a proton and anti-proton move towards each other at equal & opposite speed v

proton mass is $m_0 = 1.67 \times 10^{-27}$ kg
anti- " " is the same

$$m_0 c^2 = 1.67 \times 10^{-27} \text{ kg} * \left(3 \times 10^8 \frac{m}{s}\right)^2$$

$$= 1.503 \times 10^{-10} \text{ J} * \frac{1 \text{ eV}}{1.6 \times 10^{-19} \text{ J}} = 0.939 \times 10^9 \text{ eV}$$

$$= 939 \text{ MeV}$$

now when p & $\bar{p}$ (anti-proton) hit their entire energy gets turned into another particle called the X-particle which has a mass of

$$m_{0x} = 5.7 \text{ GeV}$$

what is the energy of the p & $\bar{p}$?

Before: $E_{tot} = E_p + E_{\bar{p}} = \gamma m_0 c^2 + \gamma m_0 c^2$
$= 2\gamma m_0 c^2$

$\vec{p}_{tot} = \vec{p}_p - \vec{p}_{\bar{p}} = 0$ opposite directions

After: $E_0 = m_{0x} c^2$ not moving

$$E_{0x} = m_{0x} c^2 = 5.7 \text{ GeV} = 2\gamma \underbrace{m_0 c^2}_{p} = 2\gamma E_{0p}$$

so $\gamma = E_{0x} / 2E_{0p}$

$E_{0x} = 5.7\,GeV$

$E_{0p} = 939\,MeV = 0.939\,GeV$

so $\gamma = \frac{5.7}{2 * 0.939} = 3.04$

so $E_p = E_{\bar{p}} = \gamma E_0 = 3.04 \times 0.939\,GeV = 2.85\,GeV$

velocity of protons: $\gamma = \frac{1}{\sqrt{1-\beta^2}} = 3.04$

$$1-\beta^2 = \frac{1}{(3.04)^2} = 0.108$$

$$\beta^2 = 1 - 0.108 = 0.89$$

$$\beta = 0.944$$

$$v = \beta c = 0.944 * 3\times10^8 \frac{m}{s}$$

$$= 2.8\times10^8 \frac{m}{s}$$

ex: 2 protons w/ equal & opposite velocity collide when they come to rest, energy is converted to a pion w/ $E_0 = m_0 c^2 = 135\,MeV$
what is initial proton velocity?

BEFORE: $E = 2E_p = 2\gamma E_{0p}$

FINAL : $E = 2E_{0p} + E_{0\pi}$

so $$2\gamma m_{op}c^2 = 2m_{op}c^2 + m_{o\pi}c^2$$

$$2\gamma m_p = 2m_p + m_\pi$$

$$\gamma = \frac{2m_p + m_\pi}{2m_p} = 1 + \frac{m_\pi}{2m_p}$$

$$= 1 + \frac{135}{2\cdot 939} \simeq 1.072$$

$$\gamma^2 = \frac{1}{1-\beta^2} \qquad (1-\beta^2)\gamma^2 = 1$$

$$\beta^2\gamma^2 = \gamma^2 - 1$$

$$\beta^2 = \frac{\gamma^2-1}{\gamma^2}$$

so $$\beta = \sqrt{\frac{(1.072)^2-1}{(1.072)^2}} \simeq 0.36$$

$$v = \beta c = 0.36 \times 3\times10^8 \frac{m}{s} = 1.08\times10^8 \frac{m}{s}$$